MONTE CARLO CALCULATION OF THE MULTIPLE SCATTERING OF MUONS

by

B.A. Chartres

School of Physics, University of Sydney, Sydney, N.S.W.

INTRODUCTION

Experimental determinations of the scattering of cosmic ray mu-mesons have yielded results in disagreement with the prediction of theory. These experiments are performed by placing layers of lead, iron or some other substance, in a cloud chamber underground and observing the angular deflections of muons as they pass through these plates.

What is observed is that the great majority of the particles experience very small deflections, of the order of one or two degrees, but a small number experience deflections of a greater order of magnitude, ranging up to 10 or 15^{0}. The proportion of the mesons which experience these large angle deflections has been found to be considerably greater than that predicted by theory.

In the course of its passage through a 1cm plate of, say, lead, a meson will undergo a number of collisions of the order of 10^{5}. Hence the theoretical prediction of the distributions of scattered angles involves a two stage calculation. The first stage is the estimation of the distribution function for the separate deflections produced by each collision, i.e. the scattering cross-section for interaction of a muon with an atomic nucleus. The second stage is the calculation of the distribution of final angles which are produced by many of these deflections. It can be seen that a valid comparison of experiment and the theory of the muon-nucleus interaction is dependent upon an accurate evaluation of the final angular distribution from a theoretically determined cross-section. This paper describes a method of performing this calculation which is at present being developed on SILLIAC.

STATEMENT OF THE PROBLEM

We shall denote by θ the angle which is measured in the experiment. This is the angle between the initial direction of the muon before entering the scatterer and its final angle as it leaves the scatterer. If photographs of the particle tracks in the cloud chamber are taken with a stereoscopic pair of cameras the actual angle θ can be measured. If a single camera is used the angle seen is θ_x, the projection of θ upon a vertical plane. Let us denote by θ_y the projection of θ upon a second, perpendicular, vertical plane.

In each collision suffered by the particle it is deflected through an angle θ_i $(i = 1,2,\ldots)$ with projections θ_{ix}, θ_{iy}. Then, because all angles involved are small, the final angle θ is determined quite accurately by

$$\theta = (\theta_x^2 + \theta_y^2)^{1/2}$$

where

$$\theta_x = \sum_i \theta_{ix}, \quad \theta_y = \sum_i \theta_{iy} \quad . \qquad \ldots.(1)$$

We denote the density distribution function for a collision which produces a deflection angle θ by $\sigma(\theta)$. Thus $\sigma(\theta)$ is the scattering cross-section correctly normalised to the scatterer thickness under consideration. We can write

$$\sigma(\theta) = \text{const.}\ 1/\theta^3.h(\theta) \qquad \ldots.(2)$$

where $h(\theta)$, known as the "form factor", is equal to unity over most of the range of θ, and falls to zero towards both ends of the range.

Due to the sharp peaking of $\sigma(\theta)$ at small values of θ the great majority of the individual collisions produce deflections which are much smaller than the r.m.s. angle of the final distribution. Consequently these deflections need not be considered in detail, as they can with great accuracy be assumed to combine to yield a gaussian distribution. We therefore cut off the function $\sigma(\theta)$ at some small angle θ_o, and allow for all the collisions which produce deflections less than θ_o by assuming that their combined result is a gaussian whose r.m.s. is equal to the r.m.s. of that part of $\sigma(\theta)$ which has been eliminated.

It is a fortunate accident, due to the particular shape of the cross-section $\sigma(\theta)$, that the average number of collisions which produce deflections greater than θ_o (we shall denote this number by Ω) is quite small. Depending upon the value of θ_o considered adequate, Ω can lie anywhere in in the range 1/2 to 5.

The general shape of the final angular distribution is easily determined. At angles well out in the "tail" the distribution is a duplicate of the collision cross-section; for such large angles are more likely to be obtained by a single large angle collision than by two or more smaller ones. At small angles the distribution may be closely approximated by a gaussian of r.m.s. somewhat greater than that of the gaussian which is obtained from the "small angle" events alone. It is in the region where these two curves join that we must perform a complete calculation to obtain any information, and for thickness of the order of 1cm this is the region in which most interest lies.

ANALYTIC SOLUTION

An analytic solution of the final angular distribution function $F(\theta)$ can easily be found. It is

$$\left.\begin{aligned} F(\theta) &= \int_{-\Omega}^{\infty} g(\theta - \theta')f(\theta')\, d\theta' \\ f(\theta) &= \sum_{n=0} f_n(\theta) \\ f_n(\theta) &= (1/n) \int \sigma(\theta - \theta')f_{n-1}(\theta')\, d\theta' \\ f_o(\theta) &= \delta(\theta)e^{-\Omega} \end{aligned}\right\} \qquad \ldots.(3)$$

where $g(\theta)$ is the gaussian resulting from the small angle deflections, $f(\theta)$ is the angular distribution resulting from the large angle deflections, Ω is the average number of large angle collisions suffered by each particle, and $\sigma(\theta)$ is the scattering probability for these collisions.

As we need not consider more than about 4 or 5 collisions for each particle the above expressions would be rather easily evaluated if θ were a scalar variable. If one were interested only in the projected angle θ_x, and not the total angle θ, then the above integrals would be easily evaluated numerically. But we are interested in the total angle of scatter θ which is a two dimensional vector. This makes the integrals in Equation (3) double integrals.

It therefore seemed likely that a Monte Carlo evaluation of these expressions was justified. The choice of the Monte Carlo method was further influenced by the fact that it is intented to expand this calculation later to cover the case of electron-photon cascade production with multiple scattering of the electrons. In this latter case the integrals become triple integrals.

THE MONTE CARLO METHOD

A Monte Carlo evaluation of the integrals in Equation (3) can be performed by setting up a model of the physical process itself. This model differs from the true physical picture only in that each particle suffers a single deflection through an angle chosen at random from the gaussian distribution $g(\theta)$ in place of the myriad small angle collisions which actually occur. We shall see later how a variation of the model, taking it further away from the actual physical picture, makes the Monte Carlo process much more efficient.

We do not try to evaluate each f_n separately, nor even the total $f(\theta)$, but go straight to $F(\theta)$. The Monte Carlo process is as follows. Each "particle" is assigned a certain number of collisions 'n' selected at random from the Poisson distribution

$$P(n) = e^{-\Omega}\Omega^n/n'. \qquad \ldots.(4)$$

We then determine the "history" of the particle, which consists of n deflections through angles θ_i chosen from the distribution $\sigma(\theta)$, and one deflection through an angle θ_g from the gaussian $g(\theta)$. By the (random) decomposition of these angles into components θ_{ix} and θ_{iy} and the addition of these components we determine the final angle θ and its components θ_x and θ_y. These angles are then entered as unit entries into the relevant intervals of two histograms. Each particle yields one entry in the histogram of total angles and two entries in the histogram of projected angles.

SELECTION OF RANDOM VARIABLES

1. The random numbers

As in all Monte Carlo calculations the selection of random variates starts with a series of pseudo-random numbers which have a rectangular distribution within the range 0 to 1. Such numbers are denoted in the following by the symbols u and v.

These random numbers were generated on SILLIAC by the method of squaring each number in the sequence and extracting the middle 38 binary digits to form the succeeding number. This method yields a sequence of about 700 000 numbers after which it degenerates to a sequence of zeros.

2. The number of collisions

We use the fact that if $u_1, u_2, u_3, \ldots$ is a sequence of random numbers, and n is chosen such that

$$\prod_{i=1}^{n} u_i > e^{-\Omega}, \quad \prod_{i=1}^{n+1} u_i \leq e^{-\Omega} \qquad \ldots.(5)$$

then the distribution of n is as defined in Equation (4).
Note that if we define

$$I_i = -(1/\Omega)\, \mathrm{Inu}_i \qquad \ldots.(6)$$

then I has the distribution

$$p(I) = \Omega e^{-I\Omega} \qquad \ldots.(7)$$

and Equation (5) is equivalent to

$$\sum_{i=1}^{n} I_i < 1, \quad \sum_{i=1}^{n+1} I_i \geq 1 \qquad \ldots.(8)$$

Thus the I_i define the actual free paths between successive collisions. For the purpose of the present calculation we do not need the values of the I_i.

3. The deflection angles θ_i

The distribution of the angles θ_i is $\sigma(\theta)$ as defined in Equation (2). As the form factor $h(\theta)$ is never greater than unity, and is equal to unity over most of the range of θ, a very economical way of sampling from $\sigma(\theta)$ is to select a θ from the distribution $1/\theta^3$, to select a random number v, and to accept or reject θ according as $h(v)$ is greater or less than v.

The selection of θ from a $1/\theta^3$ distribution is easily done by taking a pair of random numbers u_1 and u_2, and defining θ to be the reciprocal of the larger of the two. The comparison of v with $h(\theta)$ can absorb most of the time of the selection of θ, as the calculation of $h(\theta)$ occupies some 30 milliseconds. This time has been greatly reduced by storing in SILLIAC a table of values of $h(\theta)$ at 16 equally spaced points θ_k. After the initial selection of a θ we look for the two table entries $h(\theta_k)$ and $h(\theta_{k+1})$, where $\theta_k < \theta \leq \theta_{k+1}$. Then v is first compared with these two values of h. If v is less than both we reject. Only in the infrequent occurrence of v lying between $h(\theta_k)$ and $h(\theta_{k+1})$ in magnitude do we need to compute $h(\theta)$ itself.

4. Projections θ_{ix}, θ_{iy} of θ_i

To select the projections θ_{ix} and θ_{iy} of θ_i at random we need to find the sine and cosine of an angle ϕ selected at random from a rectangular distribution in the range 0 to 2π. Although there is a simple method of selecting random sines and cosines which consists of

selecting a pair of random numbers u and v which satisfy the inequality $u^2 + v^2 < 1$ and defining

$$\cos \phi = \frac{u^2 - v^2}{u^2 + v^2}$$

$$\sin \phi = \frac{2uv}{u^2 + v^2} \qquad \ldots\ldots(10)$$

it was found preferable, mainly because of the limited supply of random numbers, to store a table of cosines with 16 entries in the range 0 to $\pi/2$, and to select a sine and cosine from this table, together with appropriate signs, according to the initial six binary digits of a random number. The discreteness of the values of ϕ obtained in this way has no appreciable effect because it is superimposed on the continuous distribution of θ.

5. The gaussian angle θ_g

The distribution function from which we must sample is

$$g(\theta)\, d\theta = 2/\sigma^2 . e^{-\theta^2/\sigma^2} . \theta\, d\theta \qquad \ldots\ldots(11)$$

where σ is the r.m.s. angle. As this function has a simple indefinite integral, it is very easy to select random angles from it. We simply solve

$$u = \int_{\theta_g}^{\infty} g(\theta')\, d\theta' \qquad \ldots\ldots(12)$$

i.e.

$$\theta_g = \sigma\sqrt{-\ln u} \qquad \ldots\ldots(13)$$

We compute the projections θ_{gx} and θ_{gy} in the same way for θ_g as for θ_i. It is interesting to note that θ_{gx} and θ_{gy} are <u>independent</u> random normal variates from the normal distribution of variance $1/2 . \sigma^2$ and mean zero.

USE OF IMPORTANCE SAMPLING

The Monte Carlo calculation described above builds up a histogram which, as the number of "particles" increases, approaches the true angular distribution function. Owing to the nature of the Monte Carlo process the histogram must cover all possible values of the final angle even though we are mainly interested in the larger angles. Furthermore, owning to the natural behaviour of the real mesons, for every particle that has a large angular deflection (and therefore contributes to the "tail" of the histogram) several thousand particles go through with very small angles.

This property of the scattering distribution is found equally annoying by the experimental physicist, as he is forced to photograph the cloud chamber tracks of a thousand or so mesons before observing one with the interesting large angle scatter. However as we, unlike the experimentalist, have control over the behaviour of our "particles" we are not required to suffer from the same annoyance.

An estimate of the effect of this feature was obtained from a 3 hour run on SILLIAC, using the model described above, which yielded final angles for a total of 150 000 particles. There were about 10 000 or so entries in each of a few histogram intervals near the peak of the distribution, giving a very great freedom from the effects of random fluctuations, and hence a high accuracy; but in the interesting tail of the distribution each interval had only 3 to 10 entries.

It is apparent that what we want is a model in which the distribution of final angles is approximately rectangular, for then we shall have equal accuracy in each histogram interval. Such a model is capable of yielding the true physical distribution if to each "particle" is assigned a weight w which indicates, in some way, the probability that a real mu-meson would behave in the same way as this particle. When the final angle of each particle is found, it is entered in the appropriate interval of the histogram, not as a unit entry, but as an amount w. The histogram obtained in this way is the same shape as the one obtained by the simpler model, but, as the same number of particles have contributed to each interval, it will exhibit much smaller fluctuations in the tail.

As each particle undergoes only a small number of collissions, we can obtain a very nearly rectangular final distribution by allowing exactly one (say the first one) of the collisions suffered by each particle to yield an angle selected at random from a rectangular distribution. The effect of the following collisions will cause this rectangular distribution to be rounded off at the corners, but will not change its general shape. Thus the first deflection θ_1 is selected from a rectangular distribution, the following deflections θ_i, $i = 2,3,\ldots\ldots$, are selected from the distribution $\sigma(\theta)$.

The weight for each particle is

$$w = \sigma(\theta_1) \qquad \ldots\ldots(14)$$

With 30 divisions in the histogram, and the use of this model, a run of 6 000 particles yields 200 entries in each interval, a figure which is quite sufficient to reduce random fluctuations to a small level. Allowing a little extra time for each particle, for $\sigma(\theta)$ must now be computed at least once for every particle, a run of 6 000 particles will occupy 10 min machine time. As an equivalent accuracy in the tail would require a run of 10^7 particles under the old model we have gained a factor of roughly 1 000 to 1 in running time.

A further, though smaller, reduction in machine time can still be obtained by the use of stratified sampling. Owing to the fact that what happens to each particle after its first collision is independent of what happens in the first collision, it is not necessary that the first deflection should be chosen at random. By grouping the 6 000 particles into, say, 200 groups each consisting of 30 consecutive particles, and forcing all the particles in each group to suffer the same predetermined deflection in the first collision we can reduce the number of times that the function $h(\theta)$ must be evaluated by a factor of almost 30.